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# On The Binary Quadratic Diophantine Equation $x^2 - 7xy + y^2 + 5x = 0$

**M.A.Gopalan, S.Vidhyalakshmi, D.Maheswari***
Department of Mathematics, Shrimati Indira Gandhi College, India
matmahes@gmail.com

## Abstract

The binary quadratic equation $x^2 - 7xy + y^2 + 5x = 0$ representing hyperbola is considered and analysed for its integer points. A few interesting relations satisfied by x and y are exhibited.

**Keywords**: binary quadratic, hyperbola, integer solutions
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## Introduction

The binary quadratic equation offers an unlimited field for research because of their variety [1-5]. In this context, one may also refer [6-21]. This communication concerns with yet another interesting binary quadratic equation $x^2 - 7xy + y^2 + 5x = 0$ for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions are presented.

## Method of analysis

The hyperbola under consideration is

$$x^2 - 7xy + y^2 + 5x = 0 \qquad (1)$$

To start with, it is seen that (1) is satisfied by the following pairs of integers (1, 1), (-5, 0), (36, 6), (-245, -35). However we have other choices of solutions satisfying (1) and they are illustrated below:

**Choice 1:**

Considering (1) as a quadratic in x and solving for x, we get

$$x = \frac{1}{2}\left[7y - 5 \pm \sqrt{45y^2 - 70y + 25}\right] \qquad (2)$$

Let $\alpha^2 = 45y^2 - 70y + 25$ (3)

Substituting $\alpha = \frac{\beta}{3}$ and $y = \frac{Y+7}{9}$ (4)

in (3), we have $\beta^2 = 5Y^2 - 20$ (5)

whose least positive solution is $\beta_0 = 5, Y_0 = 3$. Considering the solutions $(\widetilde{\beta_n}, \widetilde{Y_n})$ of the pellian equation $\beta^2 = 5Y^2 + 1$ and applying Brahmagupta lemma between $(\beta_0, Y_0)$ and $(\widetilde{\beta_n}, \widetilde{Y_n})$, we have

$$\beta_{n+1} = \frac{5f}{2} + \frac{3\sqrt{5}g}{2}, \qquad Y_{n+1} = \frac{3f}{2} + \frac{\sqrt{5}g}{2} \qquad (6)$$

where $f = (9 + 4\sqrt{5})^{n+1} + (9 - 4\sqrt{5})^{n+1}$ and

$g = (9 + 4\sqrt{5})^{n+1} - (9 - 4\sqrt{5})^{n+1}$ , n = 0, 1, 2, ….

Substituting (4) and (6) in (2) and taking the positive sign, the corresponding integer solutions to (1) are given by $x_{4k+1} = F + \frac{4\sqrt{5}}{9}G + \frac{2}{9}$, $y_{4k+1} = \frac{1}{6}F + \frac{\sqrt{5}}{18}G + \frac{7}{9}$

where $F = (9 + 4\sqrt{5})^{4k+1} + (9 - 4\sqrt{5})^{4k+1}$ and

$G = (9 + 4\sqrt{5})^{4k+1} - (9 - 4\sqrt{5})^{4k+1}$ , k = 0, 1, 2, 3, ….

The recurrence relations satisfied by x and y are given by

$$x_{4k+9} = 103682x_{4k+5} - x_{4k+1} - 23040; \; x_1 = 36, \quad x_5 = 3709476$$

$$y_{4k+9} = 103682y_{4k+5} - y_{4k+1} - 80640; \; y_1 = 6, \quad y_5 = 541206$$

Some numerical examples of x and y satisfying (1) are given in the following table:

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| k | $x_{4k+1}$ | $y_{4k+1}$ |
|---|---|---|
| 0 | 36 | 6 |
| 1 | 3709476 | 541206 |
| 2 | 384605867556 | 56113239846 |
| 3 | 39876705556208676 | 5817932933091126 |

**Properties:**

1. $2860299x_{4k+1} - 27x_{4k+5} - 417312y_{4k+1} = 311040$
2. $103361x_{4k+1} - 1597547616y_{4k+5} + 233079055x_{4k+5} + 1190718720 = 0$
3. $139104x_{4k+1} - 20295y_{4k+1} - 9y_{4k+5} = 15120$
4. $105937y_{4k+5} - y_{4k+1} - 15456x_{4k+5} = 78960$
5. $98853840297x_{4k+1} - 9x_{4k+9} - 14422580928y_{4k+1} = 10749957120$

Also taking the negative sign in (2), the other set of solutions to (1) is given by

$$x_{4k+1} = \frac{1}{6}F - \frac{\sqrt{5}}{18}G + \frac{2}{9}, \qquad y_{4k+1} = \frac{1}{6}F + \frac{\sqrt{5}}{18}G + \frac{7}{9}$$

where $F = (9+4\sqrt{5})^{4k+1} + (9-4\sqrt{5})^{4k+1}$ and

$G = (9+4\sqrt{5})^{4k+1} - (9-4\sqrt{5})^{4k+1}$ , k = 0, 1, 2, 3, ….

**Properties:**

1. $15456x_{4k+1} - 105937y_{4k+1} + y_{4k+5} + 78960 = 0$
2. $40590x_{4k+1} + 18x_{4k+5} - 278208y_{4k+1} + 207360 = 0$

**Choice 2:**

Consider (1) as a quadratic in y and solve for y. Performing the analysis similar to the above choice 1, the corresponding two sets of solutions to (1) along with the properties are presented below:

**Set 1:**

$$x_{4k+3} = \frac{1}{6}F + \frac{\sqrt{5}}{18}G + \frac{2}{9}, \; y_{4k+3} = F + \frac{4\sqrt{5}}{9}G + \frac{7}{9}$$

where $F = (9+4\sqrt{5})^{4k+3} + (9-4\sqrt{5})^{4k+3}$ and

$G = (9+4\sqrt{5})^{4k+3} - (9-4\sqrt{5})^{4k+3}$ , k = 0, 1, 2, 3, ….

**Properties:**

1. $x_{4k+7} + 2255x_{4k+3} - 15456y_{4k+3} + 11520 = 0$
2. $15456x_{4k+3} - 105937y_{4k+3} + y_{4k+7} + 78960 = 0$

**Set 2:**

$$x_{4k+3} = \frac{1}{6}F + \frac{\sqrt{5}}{18}G + \frac{2}{9}, \; y_{4k+3} = \frac{1}{6}F - \frac{\sqrt{5}}{18}G + \frac{7}{9}$$

where $F = (9+4\sqrt{5})^{4k+3} + (9-4\sqrt{5})^{4k+3}$ and

$G = (9+4\sqrt{5})^{4k+3} - (9-4\sqrt{5})^{4k+3}$ , k = 0, 1, 2, 3, ….

**Properties:**

1. $x_{4k+7} - 105937x_{4k+3} + 15456y_{4k+3} + 11520 = 0$
2. $y_{4k+7} + 2255y_{4k+3} - 15456x_{4k+3} + 1680 = 0$

In addition to the above two choices of solutions, we have an another pattern as shown below:

Introducing the linear transformations $x = u + v, \; y = u - v$ (7)

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in (1), it is written as $Y^2 = 45X^2 - 20$ (8)
where $X = 2u - 1,\ Y = 18v + 5$ (9)
The smallest positive integer solution of (8) is $X_0 = 1,\ Y_0 = 5$.
Let $(\widetilde{X_n}, \widetilde{Y_n})$ be the general solution of the pellian equation $Y^2 = 45X^2 + 1$

where $\widetilde{X_n} = \frac{1}{2\sqrt{45}}\left[(161 + 24\sqrt{45})^{n+1} - (161 - 24\sqrt{45})^{n+1}\right]$

$\widetilde{Y_n} = \frac{1}{2}\left[(161 + 24\sqrt{45})^{n+1} + (161 - 24\sqrt{45})^{n+1}\right]$

Applying the lemma of Brahmagupta between the solutions $(X_0, Y_0)$ and $(\widetilde{X_n}, \widetilde{Y_n})$, the values of X and Y satisfying (8) are given by

$$X_{n+1} = X_0\widetilde{Y_n} + Y_0\widetilde{X_n}$$
$$Y_{n+1} = Y_0\widetilde{Y_n} + 45X_0\widetilde{X_n}$$

In view of (7) and (9), the values of x and y are given by

$$x_{n+1} = \frac{1}{9}\left[45\widetilde{X_n} + 7\widetilde{Y_n} + 2\right]$$
$$y_{n+1} = \frac{1}{9}\left[2\widetilde{Y_n} + 7\right]$$

Note that the value of x and y are integers when n+1 is even. Thus, the integer values of x and y satisfying (1) are represented by

$$x_{2n} = \frac{7}{18}f + \frac{5}{2\sqrt{45}}g + \frac{2}{9},\ y_{2n} = \frac{1}{9}f + \frac{7}{9}$$

where $f = (161 + 24\sqrt{45})^{2n} + (161 - 24\sqrt{45})^{2n}$

$g = (161 + 24\sqrt{45})^{2n} - (161 - 24\sqrt{45})^{2n}$

## Conclusion

As the binary quadratic equations representing hyperbolas are rich in variety, one may consider other forms of hyperbolas and search for their non-trivial distinct integral solutions along with the corresponding properties.

## References

1. L.E.Dickson, History of Theory of numbers,Vol.2, Chelsea Publishing company, New York, 1952
2. L.J.Mordell, Diophantine Equations, Academic Press, London,196
3. Andre weil, Number Theory : An approach through history:from hammurapi to legendre \ Andre weil:Boston (Birkahasuser boston),198
4. Nigel P.Smart, the algorithmic Resolutions of Diophantine equations Cambridge university press, 199
5. Smith D.E, History of mathematics Vol.I and II , Dover publicationsNew York 195
6. Gopalan M. A, Vidyalakshmi S, and Devibala S,"On the Diophantine equation $3x^2 + xy = 14$", Acta Ciencia Indica, Vol. XXXIIIM, No.2, pg.645-646,200
7. Gopalan M.A, and Janaki.G," Observations on $Y^2 = 3X^2 + 1$", Acta ciencia Indica, Vol XXXIVM, No.2,693(2008)
8. Gopalan M.A, and Vijayalakshmi.R , "Special Pythagorean triangles generated through the integral solutions of the equation $y^2 = (K^2 + 1)x^2 + 1$ ", nAntarctica J.Math,7(5),pg.503- 507(2010)
9. Gopalan M.A, and Sivagami. B , "Observations on the integral solutions of $y^2 = 7x^2 + 1$ ", Antarctica J.Math7(3),pg.291- 296 (2010)
10. Gopalan M.A amd Vijayalakshmi.R , "Observation on the integral solutions of $y^2 = 5x^2 + 1$", Impact J.Sci.Tech, Vol.4,N0.4,125 -129,2010
11. Gopalan.M.A, and Sangeetha .G, "A remarkable observation on $y^2 = 10x^2 + 1$", Impact J.Sci.Tech., Vol.4, No.1,103-106,201

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12. Gopalan M. A, Parvathy. G, "Integral points on the Hyperbola $x^2+4xy+y^2-2x-10y+24=0$", Antarctica J.Math,7(2),pg149-155,2010
13. Gopalan M. A, Palanikumar R,"Observations on $y^2=12x^2+1$" Antarctica J.Math,8(2),pg.149-152, 201
14. Gopalan M. A, Devibala S, Vijayalakshmi R,"Integral points on the hyperbola $2x^2-3y^2=5$", American journal of Applied Mathematics and Mathematical Sciences, Vol 1, No 1, pg. 1-4, Jan-June 201
15. Gopalan M. A, Vidyalakshmi. S, Usha Rani T.R, Mallika S, "Observations on $y^2=12x^2-3$", Bessel J.Math2(3),pg. 153-158, 201
16. Gopalan M. A, Vidyalakshmi. S, Sumathi G, Lakshmi K,"Integral Points on the Hyperbola $x^2+6xy+y^2+40x+8y+40=0$", Bessel J. Math.2(3) pg. 159-164,201
17. Gopalan M. A, Geetha K, " Observations on the Hyperbola $y^2=18x^2+1$", Retell,Vol.13,No.1,pg. 81-83, Nov. 201 Gopalan M. A, Sangeetha G, Manju Somanath, "Integral points on the Hyperbola $(a+2)x^2-ay^2=4a(k-1)+2k^2$" , Indian Journal of Science, Vol.1, N0.2pg. 125-126,Dec.201
18. Gopalan M. A, Vidyalakshmi S, Kavitha A, "Observations on the Hyperbola $ax^2-(a+1)y^2=3a-1$ ", Discovery, Vol.4, N0.10, pg.22-24, April.201
19. Gopalan M. A, et al., Integer points on the Hyperbola $x^2-6xy+y^2+4x=0$" Scj.J.Eng.Tech.,Vol.2,No.1, pg.14-18, 2014
20. Dr.K.Meena, et al, "On the binary quadratic Diophantine equation $x^2-5xy+y^2+18x=0$", International Journal of Engineering Research- Online, Vol.2, NO.3, pg.9-14, 2014